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The Homoclinic Orbits in the Liénard Plane

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1. INTRODUCTION

In this paper we consider the well-known Liénard equation $x'' + f(x)x' + g(x) = 0$ or an equivalent system

$$\begin{aligned} x' &= y - F(x) \\ y' &= -g(x), \qquad ' = d/dt, \end{aligned} \tag{1.1}$$

where $F(x) = \int_0^x f(u)\, du$, $f(x)$ and $g(x)$ are continuous functions on R satisfying the classic conditions

$$F(0) = 0 \qquad \text{and} \qquad xg(x) > 0 \text{ for } x \neq 0. \tag{1.2}$$

Then $(x(t), y(t)) = (0, 0)$ is a solution of (1.1), that is, the origin 0 is the only singular point of (1.1). We also assume the regularity for $f(x)$ and $g(x)$ which ensures the existence of unique solution to the initial value problem.

Obviously, system (1.1) is a special class of the system

$$\begin{aligned} x' &= a(y) + b(x) \\ y' &= c(y) + d(x) \end{aligned} \tag{1.3}$$

under the assumptions

(I) a, c are real functions of $y \in R$; b, d are real functions of $x \in R$;

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(II) a, b, c, d are of class C^1 so that every point $P \in R^2$ belongs to a unique orbit $H(P)$ of (1.3);

(III) further, $a(0) + b(0) = c(0) + d(0) = 0$,

$$[a(y) + b(x)]^2 + [c(y) + d(x)]^2 > 0 \text{ if } x^2 + y^2 > 0,$$

so that the origin $0 = (0, 0)$ is the only singular point of (1.3).

Following R. Conti, we call (1.3), under the assumptions (I), (II), and (III), a pseudolinear system.

In this paper, we let φ_t be the flow generated by (1.3) or (1.1) and we let $\omega(P)$ ($\alpha(P)$) denote the positive (negative) limit set of the orbit $H(P)$. $H^+(P)$ ($H^-(P)$) is the positive (negative) semi-orbit passing through the point P. The only singular point 0 is said to be a positive global attractor if

$$0 = \omega(P) \qquad \text{for } P \in R^2 \tag{1.4}$$

and a positive global weak attractor if

$$0 \in \omega(P) \qquad \text{for } P \in R^2. \tag{1.5}$$

Obviously, (1.4) implies (1.5), but the converse is not true for a more general system

$$\begin{aligned} x' &= X(x, y) \\ y' &= Y(x, y). \end{aligned} \tag{1.6}$$

For example we refer to [1, p. 154].

About system (1.3), R. Conti proposes the following questions:

Question 1. Are (1.4) and (1.5) equivalent for a pseudolinear system (1.3)?

Question 2. Does (1.4) imply the stability of the zero solution for a pseudolinear system?

We easily see that the two questions are closely related to the existence of homoclinic orbits; i.e., the existence of homoclinic orbits is a necessary condition for the positive global attractor to be unstable. Recently several authors have given a negative answer to Question 2 (see [2, 3]).

The plan of the paper is the following. In Section 2, we study the maximal elliptic sector which consists of homoclinic orbits and the origin (for definitions, see Section 2). This concept is important in the discussion of the stability of zero solution. For (1.1) there exists at most one maximal elliptic sector. We also have established some necessary or sufficient conditions for the existence of maximal elliptic sector. Next, in Section 3, in order to give a negative answer to Question 1 we discuss a concrete example, which also shows that the assertion about stability in [4, Remark 3.1] is wrong.

2. Maximal Elliptic Sector

First, we introduce some notations and definitions.

For system (1.3), let $J(x, y)$ be the Jacobian matrix at point (x, y), that is,

$$J(x, y) = \begin{pmatrix} b'(x) & a'(y) \\ d'(x) & c'(y) \end{pmatrix}, \tag{2.1}$$

and denote

$$\operatorname{tr} J(x, y) = b'(x) + c'(y),$$
$$\operatorname{Det} J(x, y) = b'(x)c'(y) - a'(y)d'(x).$$

A homoclinic orbit $H(P)$ passing through $P \neq 0$ is an orbit satisfying $\omega(P) = \alpha(P) = 0 = (0, 0)$. $H(P)$ is homoclinic if $H(P)$ is a homoclinic orbit.

Because the origin 0 is the only singular point of system (1.3), by the Poincaré–Bendixson theorem we easily conclude

Lemma 2.1. *If $H(P)$ is a homoclinic orbit of* (1.3), *then every orbit passing through a point in the region D, which is surrounded by $H(P) \cup 0$, is a homoclinic orbit.*

Lemma 2.2. *If there exists a non-trivial whole orbit of* (1.3) *in any neighborhood of the origin, then* $\operatorname{tr} J(0, 0) = 0$, *and* $\operatorname{Det} J(0, 0) \geq 0$.

Proof. For a positive number $r > 0$, let $B(r) = \{(x, y)\colon x^2 + y^2 < r^2\}$ be a neighborhood of 0. If $\operatorname{tr} J(0, 0) = 2e \neq 0$, without loss of generality, we assume $e > 0$. The continuity of $\operatorname{tr} J(x, y)$ implies that there exists an $r_0 > 0$ so that

$$\operatorname{tr} J(x, y) \geq e > 0 \qquad \text{for } (x, y) \in B(r_0). \tag{2.2}$$

If $H(P)$ is a whole non-trivial orbit (i.e., $H \neq 0$) in $B(r_0)$, then $\omega(P) \subset \overline{B(r_0)}$. If $0 \,\overline{\in}\, \omega(P)$ the Poincaré–Bendixson theorem yields that $\omega(P) = H_1$ is a closed orbit. This contradicts the Bendixson theorem, which ensures that there exist no closed orbits in $B(r_0)$. By the same way we also have contradictions for the following two cases: (i) $0 \in \omega(P)$ and $0 \,\overline{\in}\, \alpha(P)$, (ii) $0 \in \omega(P) \cap \alpha(P)$; i.e., there is a homoclinic orbit in $B(r_0)$. So we assert tr $J(0, 0) = 0$. Now if Det $J(0, 0) < 0$, then the origin 0 is a saddle, it is impossible that there is a whole non-trivial orbit in a sufficiently small neighborhood of the origin. Q.E.D.

Remark 2.3. Apparently Lemma 2.2 is true for the more general C^1 system (1.6).

Furthermore, we have

LEMMA 2.4. *If there exists a homoclinic orbit in any neighborhood of the origin for* (1, 3), *then* $b'(0) = c'(0) = 0$, and Det $J(0, 0) = 0$.

Proof. By Lemma 2.2 we obtain tr $J(0, 0) = 0$. Now if Det $J(0, 0) > 0$, then the origin is a center or a focus or a center focus (see [6, Theorem 5.1, p. 102]), this is a contradiction of the assumption of the lemma and yields Det $J(0, 0) = 0$. If $b'(0) \neq 0$, let $b'(0) = \alpha > 0$ (for $b'(0) < 0$, replace t by $-t$ in (1.3)). Then tr $J(0, 0) = 0$ implies $c'(0) = -\alpha$, and denote $a'(0) = \beta \neq 0$, Det $J(0, 0) = 0$ leads to $d'(0) = -\alpha^2/\beta$. In a sufficiently small neighborhood $B(r_1)$ of 0, system (1.3) is equivalent to the system

$$\begin{aligned} x' &= \alpha x + \beta y + o(\rho) \\ y' &= -\tfrac{\alpha}{\beta}(\alpha x + \beta y) + o(\rho), \end{aligned} \tag{2.3}$$

where $\rho = [x^2 + y^2]^{1/2}$.

Thus we conclude that if $(x, y) \in B(r_1)$ and r_1 (>0) small enough, dy/dx is of the same sign with $-\alpha/\beta$ or $dy/dx = 0$, which leads to the monotonicity of the solution curves in $B(r_1)$. This is contradictory to the existence of homoclinic orbits in $B(r_1)$. Q.E.D.

For further discussion we need

DEFINITION 2.5. For homoclinic orbits $H(P_1)$ and $H(P_2)$, if $H(P_1)$ is contained in the region surrounded by $H(P_2) \cup 0$, we call $H(P_1)$ and $H(P_2)$ in the same class. By a maximal elliptic sector we mean the closure of the region consisting of all the homoclinic orbits in the same class.

Remark. For the concept of elliptic sector we refer to [7, p. 163]. In the sequel we focus on the Liénard system

$$x' = y - F(x), \qquad y' = -g(x), \tag{2.4}$$

where $F(x) = \int_0^x f(u)\, du$, $f(x)$ and $g(x)$ are continuous, and $xg(x) > 0$ for $x \neq 0$.

PROPOSITION 2.6. *System* (2.4) *has at most one maximal elliptic sector.*

Proof. Let $(x(t), y(t))$ be a solution of (2.4), the assumption $xg(x) > 0$ guarantees that $y(t)$ is monotone increasing in the left half plane $D_1 =$

$\{(x, y)\colon x < 0\}$, and monotone decreasing in the right half plane $D_2 = \{(x, y)\colon x > 0\}$ when t increases. Then it is easy to see that any homoclinic orbit crosses the y-axis. Let $P = (0, y_0)$ be a point on the y-axis, and $H(P)$ is a homoclinic orbit. Suppose $y_0 > 0$, we conclude that $H^-(P) \subset D_1$ and $\varphi_t(P) \to 0$ $(t \to -\infty)$, otherwise, $H^-(P)$ intersects the negative y-axis at point Q since $H(P)$ is a homoclinic orbit. We consider two cases:

Case 1. $H^-(Q) \subset D_2$ and $\varphi_t(Q) \to 0 (t \to -\infty)$. In the right half-plane D_2, if the solution $(x(t), y(t))$ is under the vertical isocline $y = F(x)$ then $x(t)$ is increasing, so $\varphi_t(Q) \to 0$ $(t \to -\infty)$ implies that $H^-(Q)$ intersects $y = F(x)$ at R. By the boundedness of $H^+(P)$, $H^+(P)$ also intersects $y = F(x)$ at W, which is a contradiction of the fact that $H(P)$ is a homoclinic orbit.

Case 2. $H^-(Q)$ intersects the positive y-axis at R. Let D be the region surrounded by solution arc RQP and segment $\overline{\mathrm{PR}}$. Then $H^+(P) \cap D = \varnothing$ or $H^-(P) \cap D = \varnothing$. This contradicts $\omega(P) = 0$ or $\alpha(P) = 0$.

Similarly, $H^+(P) \subset D_2$ and $\varphi_t(P) \to 0$ $(t \to \infty)$. Now $H(P) \cup 0$ consists of a simple closed curve. Lemma 2.1 leads to the conclusion that there exists $P_1 = (0, y_1)$ (y_1 may be ∞) on the positive y-axis, if $y_1 < \infty$, then $H(P_1)$ is a homoclinic orbit and also $H(P)$ $(P = o, y))$ for $0 < y < y_1$, but for $y > y_1$ $H(P)$ is not homoclinic. If $y_1 = \infty$, then every orbit crossing the positive y-axis is homoclinic.

The above discussion concludes that there exists at most one maximal elliptic sector S_1 for the positive y-axis and also one S_2 for the negative y-axis. If S_1 and S_2 do exist, then $S_1 \cap S_2 = \{0\}$.

If $H(P)$ is homoclinic, let $P_i = (x_i, y_i) \in H(P)$ $(i = 1, 2)$, where $x_1 = \max\{x\colon (x, y) \in H(P)\} > 0$ and $x_2 = \min\{x\colon (x, y) \in H(P)\} < 0$, and $y_i = F(x_i)$. Obviously, $H(P)$ crosses $y = F(x)$ at P_i, which means P_i belongs to the graph of $y = F(x)$. That is to say, every maximal elliptic sector contains an arc of $y = F(x)$. Then there exists a $\delta > 0$ so that the arc $G_\delta = (x, y)\colon |x| < \delta,\ y = F(x)\} \subset S_1 \cap S_2$. This contradicts $S_1 \cap S_2 = \{0\}$. Thus, system (2.4) has at most one maximal elliptic sector S. The proof is complete. Q.E.D.

PROPOSITION 2.7. *If there exists a homoclinic orbit $H(P)$ of system* (2.4), *then $F'(0) = F(0) = 0$ and there exists a positive $\delta > 0$ such that $F(x) \neq 0$ for $0 < |x| < \delta$, and $y = F(x)$ does not traverse the x-axis at the origin; that is, the vertical isocline is tangent to the x-axis in even order at the origin.*

Proof. By Lemma 2.4 $F'(0) = 0$. Suppose that P is on the positive half y-axis. Denote by D the region surrounded by $H(P) \cup 0$. From the proof of Proposition 2.6, define $\delta = \min\{x_1, -x_2\} > 0$. Thus, the arc $G_\delta = \{(x, y)\colon |x| < \delta,\ y = F(x)\} \subset D$. Suppose there exists an x^* $(0 <$

$|x^*| < \delta$) so that $F(x^*) = 0$. Suppose that $x^* < 0$, let $Q = (x^*, 0) \in G_\delta$, now Lemma 2.1 guarantees that $H(Q)$ is homoclinic. By the monotonicity of $y(t)$ for the solution $(x(t), y(t))$ in the left half-plane D_1, $H^-(Q)$ traverses the negative y-axis, which implies that $H(Q)$ is not homoclinic (see the proof of Proposition 2.6). For $x^* > 0$ the proof is similar. Thus, $F(x) \neq 0$ for $0 < |x| < \delta$.

Next suppose that $y = F(x)$ traverses the x-axis at the origin (i.e., tangent in odd order). We deal only with the case $F(x) < 0$ for $0 < x < \delta$ and $F(x) > 0$ for $-\delta < x < 0$. The other case is similar.

For $G_\delta \subset D$, the boundedness and monotonicity of $x(t)$, $y(t)$ in various regions guarantee that $H(P)$ intersects $y = F(x)$ at a point $Q^* = (x', F(x'))$, $x' > 0$. Since $F(x') < 0$, $H^+(Q^*)$ intersects the negative half y-axis, which contradicts the fact that $H(P)$ is homoclinic. Thus, we assert that $y = F(x)$ is tangent to the x-axis in even order at the origin.

Q.E.D.

THEOREM 2.8. *If every positive semi-orbit passing through a point in a neighborhood of the origin is bounded and there exist no non-trivial closed orbits for system* (2.4), *then the existence of homoclinic orbits is equivalent to the fact that there exists $d > 0$ such that $F(x) \neq 0$ $(0 < |x| < d)$ and*

1. *For $F(x) > 0$ $(0 < |x| < d)$, there exists a point $P \in D^+ = \{(x, y): -d < x < 0$ and $0 < y < F(x)\}$ so that $H^-(P) \subset D^+$ and $\varphi_t(P) \to 0$ $(t \to -\infty)$.*

2. *For $F(x) < 0$ $(0 < |x| < d)$, there exists a point $P \in D^- = \{(x, y): 0 < x < d$ and $F(x) < y < 0\}$ so that $H^-(P) \subset D^-$ and $\varphi_t(P) \to 0$ $(t \to -\infty)$.*

Proof. We shall prove only the case $F(x) > 0$ $(0 < |x| < d)$. For the other case the proof is similar.

Since $\alpha(P) = 0$, we need to prove $\omega(P) = 0$. Choose $\delta > 0$ and $H^+(P)$ is bounded. Thus, $H^+(P)$ intersects $y = F(x)$, the positive y-axis, and $y = F(x)$ in that order. Suppose that $H^+(P)$ traverses the positive y-axis at Q. If $H^+(Q) \subset D_2$, then $\omega(Q) = 0$ from the boundedness and monotonicity of $H^+(Q)$. So $\omega(P) = 0$. Otherwise, $H^+(Q)$ traverses the negative y-axis at R. Thus, $H^+(R)$ lies outside of the region surrounded by $H^-(R) \cup \overline{OR}$. The boundedness of $H^+(R)$ implies the existence of a non-trivial closed orbit. This is a contradiction.

Conversely, we derive the conclusion from the proofs of Propositions 2.6 and 2.7. Q.E.D.

For application, we have

COROLLARY 2.9. *Suppose that every positive semi-orbit passing through a point in a neighborhood of the origin is bounded and that there* *exist no non-trivial closed orbits for system* (2.4). *Furthermore, suppose that there exist $d > 0$ and a positive C^1 function $h(x) > 0$ $(x \neq 0)$, $h(0) = 0$ such that*

$$\begin{aligned} &F(x) > 0 \qquad \text{for } 0 < |x| < d, \\ &g(x) > h(x)(F'(x) - h'(x)) \qquad \text{for } -e < x < 0 \end{aligned} \tag{2.5}$$

where $e > 0$ is sufficiently small. Then system (2.4) *has homoclinic orbits.*

Proof. Consider the region $D^* = \{(x, y){:}\ -e < x < 0$ and $F(x) - h(x) < y < F(x)\}$, (2.5) implies that D^* is negatively invariant for the flow φ_t generated by system (2.4). Hence, there exists a $P \in D^*$ so that $H^-(P) \in D^*$ and $\varphi_t(P) \to 0$ $(t \to -\infty)$. Then the corollary follows from Theorem 2.8. (Q.E.D.)

COROLLARY 2.10. *Under the same assumptions as in Corollary* 2.9 *and under the assumption that $F(x) < 0$ for $0 < |x| < d$ and that there exists a positive C^1 function $h(x) > 0$ $(x \neq 0)$, $h(0) = 0$ so that*

$$-g(x) > h(x)(F'(x) + h'(x)) \qquad \text{for } 0 < x < e, \tag{2.6}$$

where $e > 0$ is small enough, it follows that system (2.4) *has homoclinic orbits.*

3. WEAK ATTRACTOR

If all the solutions $(x(t), y(t))$ $(t \geq 0)$ of (1, 1) are bounded, then for any $P \in R^2$, the positive semi-orbit $H^+(P) \subset B(r)$ for some $r > 0$, and $\omega(P)$ is a non-empty closed set in $B(r)$. If $0 \overline{\in} \omega(P)$, by the Poincaré–Bendixson theorem $\omega(P)$ is a closed orbit. Hence, the absence of non-trivial closed orbits of (1, 1) implies $0 \in \omega(P)$; i.e., 0 is a positive global weak attractor. However, an attractor may be unstable. Several authors [2, 3] have established examples to show that the origin 0 is an attractor but unstable. This implies that the conclusion in [4, Remark 3.1] is wrong (also see the following Proposition 3.3). In the sequel, we employ an example to give a negative answer to Question 1.

For system (1.1), let $M^+ = \int_0^\infty g(u)\,du$, $M^- = \int_0^{-\infty} g(u)\,du$, and $M = \min\{M^+, M^-\}$. We remark that M^+ and M^- are positive and may be ∞. Define $w = G(x) = \int_0^x |g(x)|\,du$, then by (1.2), $G(x)$ is strictly increasing, and hence there exists $G^{-1}(w)$, the inverse function of $G(x)$, for $|w| < M$. The following lemma is the main result of [4, Theorem 3.1].

LEMMA 3.1. *Suppose that*

$$F(G^{-1}(-w)) \neq F(G^{-1}(w)) \qquad \text{for } 0 < w < M.$$

Then (1.1) has no periodic solutions except for the origin.

Consider the Liénard equation $x' + x(x-1)x' + kx^3 = 0$ $(k > 0)$ (see [3]), or an equivalent system

$$x' = y - (\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2), \qquad y' = -kx^3\,(k > 0). \tag{3.1}$$

We have

PROPOSITION 3.2. *The unique singular point 0 of* (3.1) *is a global attractor but unstable for* $k \in (0, 1/8)$.

Proof. Because $F(x) \to \infty$ (resp. $-\infty$) if $x \to \infty$ (resp. $-\infty$) and $\int_0^x g(u)\,du \to \infty$ if $|x| \to \infty$, by the Mizohata–Yamaguti criterion [8, p. 111], every positive semi-orbit of (3.1) is bounded. Moreover, Lemma 3.1 ensures that (3.1) has no periodic solutions except for the origin. We assert that the origin is a global weak attractor. Choose $h(x) = \frac{1}{4}x^2$. It is easy to verify that there exists an $e > 0$ to guarantee

$$-g(x) > h(x)(F'(x) + h'(x)) \qquad \text{if } 0 < x < e,$$

thus, Proposition 3.2 follows from Corollary 2.10. Q.E.D.

Remark. A similar result is obtained in [3] by complex estimates of the orbits and the study of stability of zero solution.

In order to give a negative answer to Question 1, consider the concrete Liénard system

$$x' = y - (\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2), \qquad y' = -x^3, \tag{3.2}$$

where $F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2$ and $g(x) = x^3$. As in Proposition 3.2, choose $h(x) = \frac{3}{4}x^2$. We easily conclude

PROPOSITION 3.3. *The unique singular point* 0 *of* (3.2) *is a global attractor but unstable.*

Corresponding to system (3.2), consider a supplementary system

$$x' = y - F_0(x), \qquad y' = -x^3, \tag{3.3}$$

where

$$F_0(x) = \begin{cases} \frac{1}{3}x^3 - \frac{3}{2}x^2, & x \geq 0 \\ -\frac{1}{3}x^3 - \frac{3}{2}x^2, & x < 0. \end{cases} \tag{3.4}$$

Then $F_0(-x) = F_0(x)$, or $F_0(x)$ is an even function, which ensures that the orbits of (3.3) have mirror symmetry about the y-axis in the phase plane. For a Liénard system, it is easy to determine the directions of solution vectors on the y-axis and the curve $y = F(x)$. Furthermore, the boundedness of positive semi-orbits of (3.2) implies that every solution of (3.3) passing through a point on the positive y-axis intersects the negative y-axis. Hence, we present the phase-portraits of (3.2) and (3.3) in Figs. 3.1a and 3.1b, respectively.

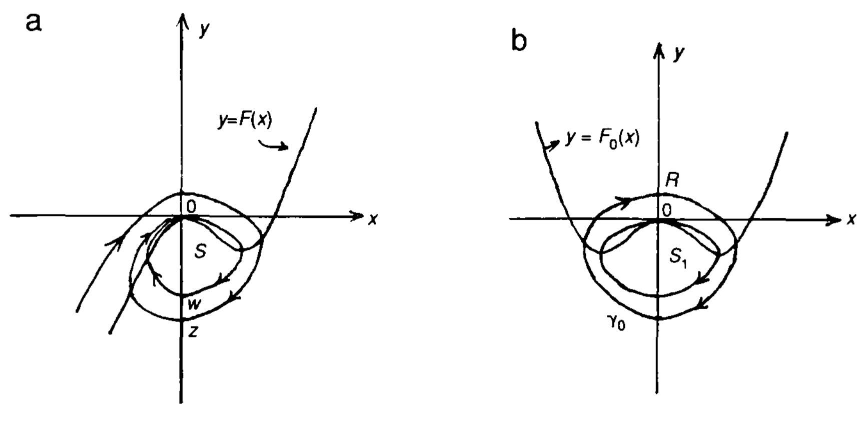
Note that the solution curves of (3.2) and (3.3) coincide in the right half-plane. Let S be the maximal elliptic sector of (3.2). Then the boundary ∂S of S must be the union of a solution curve γ of (3.2) and the origin 0. If every solution orbit outside S surrounds S, then system (3.2) is an example we need. Otherwise, suppose that S crosses the negative y-axis at W, there exists an orbit $H(Z)$ of (3.2) passing through a point Z under W on the negative y-axis, and $H^+(Z)$ lies in a sufficiently small neighborhood of S, with $H^+(Z)$ in the left half-plane and $\omega(Z) = 0$.

Define

$$F_z(x) = (1 - z)F_0(x) + zF(x), \tag{3.5}$$

where $F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2$ and $z \in [0, 1]$ and establish a new system

$$x' = y - F_z(x), \qquad y' = -x^3. \tag{3.6}$$



In what follows, we shall prove that for suitable $z > 0$ system (3.6) has the asymptotic behavior we need; that is, (1.5) holds but (1.4) does not hold.

Let S_1 be the maximal elliptic sector of (3.3). Denote by N^* the outside part of a neighborhood N of ∂S, i.e., $N^* = N \backslash S$. Consider a closed orbit γ_0 of (3.3) in N^* and let s be the arc length of γ_0 in the clockwise direction, starting from the point R at which γ_0 crosses the positive y-axis. Using the parameter s, we define $\gamma_0 = (u(s), v(s))$ and let n be the length of normal of γ_0 at $(u(s), v(s))$ whose direction is outside. Thus, we define the transformation

$$x = u(s) - nv'(s), \qquad y = v(s) + nu'(s), \tag{3.7}$$

where $p_0 = v(s) - F_0(u(s))$, $Q_0 = -u^3(s)$, and

$$u'(s) = \frac{P_0}{\sqrt{P_0^2 + Q_0^2}}, \qquad v'(s) = \frac{Q_0}{\sqrt{P_0^2 + Q_0^2}}. \tag{3.8}$$

Put

$$\frac{dy}{dx} = \frac{v'(s) + u'(s)\, dn/ds + nu''(s)}{u'(s) - v'(s)\, dn/ds - nv''(s)}$$

and (3.7) into (3.6). We have

$$\frac{dn}{ds} = \frac{Qu' - Pv' - n(Pu'' + Qv'')}{Pu' + Qv'} \overset{\text{Def}}{=} F_2(s, n, z), \tag{3.9}$$

where $n = n(s, n_0, z)$, $n_0 = n(0, n_0, z)$, and

$$P = v(s) + nu'(s) - F_z(u(s) - nv'(s)), Q = -(u(s) - nv'(s))^3.$$

For the above argument, we refer to [9, Chap. 2].

Obviously, the orbits of (3.3) in N^* are closed orbits. It is easy by (3.9) to assert that

$$n(L, n_0, 0) = n_0 + \int_0^L F_2(s, n(s, n_0, 0), 0)\, ds = n_0, \tag{3.10}$$

where L is the arc length of the closed orbit. By the discussion about the phase-portraits of (3.2), from (3.9) we obtain

$$n(L, n_0, 1) = n_0 + \int_0^L F_2(s, n(s, n_0, 1), 1)\, ds = -\lambda, \tag{3.11}$$

where λ is the length of $\overline{RO}$. Since $F_0(x) \geq F_{z_1}(x) \geq F_{z_2}(x) \geq F(x)$ for $0 < z_1 < z_2 < 1$, by the comparison theorem (or see Lemma 6.2 in [5]) it is easy to verify that

$$n(L, n_0, z_1) \geq n(L, n_0, z_2) \qquad \text{for } 1 > z_2 > z_1 > 0. \tag{3.12}$$

Now rewrite (3.9)

$$dn/ds = F_2(s, 0, z) + \left.\frac{\partial F_2}{\partial n}\right|_{n=0} \cdot n + o(n), \tag{3.13}$$

where

$$F_2(s, 0, z) = \frac{Q'P_0 - P'Q_0}{P'P_0 + Q'Q_0} \qquad \text{and} \qquad P' = P|_{n=0}, Q' = Q|_{n=0}.$$

Consider the solution of (3.13) with the initial value $(s, n) = (0, n_0)$. Define

$$I_z = \int_0^L F_2(s, 0, z)\, ds. \tag{3.14}$$

Since $F(x) = F_z(x) = F_0(x)$ for $x > 0$, it follows from the symmetry of γ_0 about the y-axis that

$$I_z = \int_{L/2}^L F_2(s, 0, z)\, ds. \tag{3.15}$$

Hence, for $n_0 = 0$ (3.10) and (3.11) correspond to

$$I_0 = 0 \qquad \text{and} \qquad I_1 = -\lambda + 0(n_0). \tag{3.16}$$

Let $N_1 = \{(x, y)\colon (x, y) \in N^* \text{ and } x \leq 0\}$ and $N_2 = \{(x, y)\colon (x, y) \in N^*$ and $F(x) \leq y \leq F_0(x)\}$; define

$$H_z(x, y) = \frac{\left(y + \frac{1}{3}x^3 + \frac{3}{2}x^2\right)\left(y - \frac{1}{3}x^3 + \frac{3}{2}x^2\right) + x^6}{\left(y + \frac{1}{3}x^3 + \frac{3}{2}x^2\right)\left(y - \frac{2z-1}{3}x^3 + \frac{3}{2}x^2\right) + x^6} \tag{3.17}$$

for $(x, y) \in \overline{N_1}$, $x^2 + y^2 \neq 0$, and $H_z(0, 0) = \lim_{\substack{x\to 0\\ y\to 0}} H_z(x, y)$.

We have

LEMMA 3.4. *For $(x, y) \in \overline{N_1}$, there exist positive numbers $M > 0$ and $z_0 > 0$ such that $|H_z(x, y)| < M$ for $0 \le z \le z_0 < 1$.*

Proof. For $(x, y) \in N_2$, there exists a positive number $w > 0$ such that $|x| > w > 0$. Since

$$\begin{aligned}&\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)\left(y-\frac{2z-1}{3}x^3+\frac{3}{2}x^2\right)+x^6\\&\quad=\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)^2+x^6-\frac{2z}{3}x^3\left(y-\frac{1}{3}x^3+\frac{3}{2}x^2\right)\end{aligned}\tag{3.18}$$

and since N_2 is a closed set with $0 \overline{\in} N_2$, it follows that for sufficiently small z (i.e., $0 < z \le z_0 < 1$), the value of (3.18) is not less then $\theta > 0$. Thus, $H_z(x, y)$ is uniformly bounded on N_2. In order to get the boundedness of $H_z(x, y)$ on $\overline{N_1}$, we need only consider the case that (x, y) belongs to the part of $\overline{N_1 \backslash N_2}$ containing the origin (on the other part, $|y| \ge w^* > 0$, the boundedness is obvious). Now

$$y+\tfrac{1}{3}x^3+\tfrac{3}{2}x^2>0 \qquad \text{and} \qquad y-\tfrac{1}{3}x^3+\tfrac{3}{2}x^2>0,$$

and then

$$\begin{aligned}H_z(x,y) &\le \frac{\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)\left(y-\frac{1}{3}x^3+\frac{3}{2}x^2\right)+x^6}{\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)^2+x^6}\\&=\frac{\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)^2-\frac{2}{3}x^3\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)+x^6}{\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)^2+x^6}\\&\le 1+\frac{-\frac{2}{3}x^3\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)}{2\left(y+\frac{1}{3}x^3+\frac{3}{2}x^2\right)|x^3|}\doteq\frac{4}{3}.\end{aligned}\tag{3.19}$$

This completes the proof. Q.E.D.

From (3.15), by the mean value theorem, we obtain

$$I_z = \int_{L/2}^{L} \frac{-\frac{2}{3} z x^6 \, ds}{\left(y + \frac{1}{3} x^3 + \frac{3}{2} x^2\right)\left(y - \frac{2z-1}{3} x^3 + \frac{3}{2} x^2\right) + x^6}$$

$$= z \int_{L/2}^{L} \frac{-\frac{2}{3} x^6 H_z(x, y) \, ds}{\left(y + \frac{3}{2} x^2\right)^2 + \frac{8}{9} x^6} \qquad \text{(mean value theorem)} \tag{3.20}$$

$$= z\psi(z) \int_{L/2}^{L} \frac{-\frac{2}{3} x^6 \, ds}{\left(y + \frac{3}{2} x^2\right)^2 + \frac{8}{9} x^6}$$

$$= z\psi(z) I_1,$$

where $x = u(s)$, $y = v(s)$, and $\psi(z) = H_z(u(s'), v(s'))$ for some $s' \in [L/2, L]$.

By the uniform boundedness of $H_z(x, y)$ (Lemma 3.4), there exists a positive number $E > 0$ such that $|\psi(z)| < E$. Thus, from the monotonicity of $n(L, n_0, z)$ with respect to z (see (3.12)), for some $z^* > 0$ we conclude that

$$0 > I_z > \tfrac{1}{3} I_1 \qquad \text{for } z \in (0, z^*]\ (0 < z^* < 1). \tag{3.21}$$

Then, by (3.13),

$$n(L, n_0, z) = n_0 + I_z + 0(n_0) = I_z + 0(n_0). \tag{3.22}$$

If the neighborhood N is sufficiently small and n_0 is also small enough, then (3.21) and (3.22) imply

$$\begin{aligned} 0 &> n(L, n_0, z) > -\tfrac{1}{3}\lambda + 0(n_0) \\ &> \tfrac{1}{2} n(L, n_0, 1) > -\lambda (z \in (0, z^*)). \end{aligned} \tag{3.23}$$

For arbitrary $n_0 \in (-\lambda, 0)$ (3.23) holds if $z \in (0, z^*)$. Hence, choose $z \in (0, z^*)$. System (3.6) then has a maximal elliptic sector S^* and every orbit outside of S^* surrounds S^* and tends to ∂S^*; that is, (1.5) holds but (1.4) does not hold. We obtain

THEOREM 3.5. *There exists a pseudolinear system with a positive global weak attractor which is not a positive global attractor; i.e.,* (1.5) *does not imply* (1.4).

Theorem 3.5 gives a negative answer to Question 1.

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